Axisymmetric Vibration of Circular Sandwich Plates

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Nomenclature

ρ, ρ_c = mass density of the face and core materials, respectively
G, G_c = modulus of rigidity of the face sheet and core, respectively
E, ν = modulus of elasticity and Poisson's ratio of the face sheet
$h, 2c$ = thickness of face sheet and core, respectively
u = radial displacement of midsurface of face sheet
w = displacement in transverse direction
β_r = rotation of the normal to the middle surface of the face sheet in the radial direction
a = radius of the sandwich plate
r = radial polar coordinate
ω = circular frequency of the sandwich plate
t = time
$u_{,r}$ = partial differentiation of u with respect to r
∇^2 = $\partial^2/\partial x^2 + (1/x)\,\partial/\partial x$

Introduction

FLEXURAL vibration of circular homogeneous plates based on classical theory has been studied quite extensively and well documented.[2,7] The effect of rotary inertia and shear deformation has been considered in homogeneous plates.[1,3] The problem of vibration of rectangular sandwich plates including the effect of rotary inertia and shear deformation has also been discussed by Yu.[5,6]

The purpose of this Note is twofold: to discuss vibration of sandwich circular plates which is not available in literature and to simplify greatly the governing differential equations to a form from which the solution can be conveniently obtained. The latter is achieved by the introduction of two functions defined by Eq. (5).

In this Note, axisymmetric vibration of three-layered circular plates is investigated. The two face sheets are assumed to be made of same isotropic material with equal thickness and the thick core layer is a low density, low strength material such as aluminum honeycomb. The face sheets are assumed to take bending and because of the nature of the type of core material considered in this paper, the face parallel stresses in the core are neglected. The core is perfectly bonded to the two face sheets and is assumed to be incompressible in the transverse direction z. The differential equations are general in nature as they include the effects of rotary inertia and transverse shear.

The numerical results presented at the end are for a clamped circular sandwich plate and show the effects of variation of c/a and h/a on the frequency ω for three modes. A graph showing modal shapes is also included.

Basic Equations

Energy technique similar to that employed for elastic shells,[4] has been used to obtain the differential equations. The face sheets are capable of taking bending stresses whereas the thick central core takes only the transverse shear. The distribution of radial displacement along the thickness is shown in Fig. 1. The variation technique is used on the energy expression for the plate element. The equations of motion (1) are obtained by the variation of parameters u, w, and β_r and the stress-resultants and displacement relations (2) are achieved by the variation of N_r, N_θ, Q_{rc}, etc.

The equations of motion for the axisymmetric vibration of circular sandwich plate are given as

$$\begin{aligned} &N_{r,r} + (N_r - N_\theta)/r - Q_{rc}/2c = \rho h(s_2 u_{,tt} + s_3 a\beta_{r,tt}) \\ &(Q_r + Q_{rc}/2)_{,r} + (Q_r + Q_{rc}/2)/r = \rho h s_1 w_{,tt} \\ &M_{r,r} + (M_r - M_\theta)/r - Q_r + Q_{rc}\, h/4c = \\ &\qquad \rho h a[s_3 u_{,tt} + \tfrac{1}{12}(h^2/a^2)s_1 a\beta_{r,tt}] \end{aligned} \tag{1}$$

The stress-resultants and displacement relationships are

$$\begin{aligned} &N_r = K(u_{,r} + \nu u/r),\quad N_\theta = K(u/r + \nu u_{,r}) \\ &M_r = D(\beta_{r,r} + \nu\beta_r/r),\quad M_\theta = D(\beta_r/r + \nu\beta_{r,r}) \\ &Q_r = \tfrac{5}{12}K(1-\nu)[\alpha_1(w_{,r} + \beta_r) + \alpha_3(w_{,r} + u/c - \beta_r h/2c)] \\ &Q_{rc} = \tfrac{5}{6}K(1-\nu)[\alpha_3(w_{,r} + \beta_r) + \alpha_4(w_{,r} + u/c - \beta_r h/2c)] \end{aligned} \tag{2}$$

In the preceding equations (1) and (2), various symbols used are defined as follows:

$$\begin{aligned} &s_1 = 1 + r_\rho r_h,\quad s_2 = 1 + r_\rho r_h/3,\quad s_3 = -r_\rho(c/a)/6 \\ &K = Eh/(1-\nu^2),\quad D = Eh^3/[12(1-\nu^2)] \\ &\alpha_1 = 1 + r_G/(15r_G + 20r_h),\quad \alpha_3 = 2r_G r_h/(15r_G + 20r_h) \\ &\alpha_4 = r_G r_h(15r_G + 24r_h)/(15r_G + 20r_h) \\ &r_\rho = \rho_c/\rho,\quad r_G = G_c/G,\quad r_h = c/h \end{aligned} \tag{3}$$

For normal modes of vibration, the three displacement and slope variables are written as

$$\begin{aligned} u &= \exp(i\omega t)U \\ a\beta_r &= \exp(i\omega t)V \\ w &= \exp(i\omega t)W;\quad i = (-1)^{1/2} \end{aligned} \tag{4}$$

In order to solve the differential equations of motion (1), new deformation functions Φ and Γ are introduced which are given by Eq. (5).

$$\begin{aligned} \Phi &= U_{,x} + U/x \\ \Gamma &= V_{,x} + V/x \end{aligned} \tag{5}$$

where

$$x = r/a$$

Substitutions of Eqs. (2, 4, 5) in Eqs. (1) reduce the form of equations of motion to

$$\begin{aligned} &\nabla^2 W + A_1 W + A_2\Phi + A_3\Gamma = 0 \\ &A_4 W_{,x} + A_5 U + \Gamma_{,x} + A_6 V = 0 \\ &A_7 W_{,x} + \Phi_{,x} + A_8 U + A_9 V = 0 \end{aligned} \tag{6}$$

The coefficients A_i, used in Eqs. (6), are given as

$$\begin{aligned} &A_1 = (12/5)(1+\nu)(\omega/\omega_0)^2(h/a)^2 s_1/\alpha_5 \\ &A_2 = (\alpha_3 + \alpha_4)(a/c)/\alpha_5 \\ &A_3 = [\alpha_1 + \alpha_3 - (\alpha_3 + \alpha_4)/2r_h]/\alpha_5 \\ &A_4 = -5(1-\nu)[\alpha_1 + \alpha_3 - (\alpha_3 + \alpha_4)/2r_h](a/h)^2 \\ &A_5 = -5(1-\nu)(\alpha_3 - \alpha_4/2r_h)(a^3/h^2 c) + 12s_3(1-\nu^2)(\omega/\omega_0)^2 \\ &A_6 = -5(1-\nu)(\alpha_1 - \alpha_3/r_h + \alpha_4/4r_h^2)(a/h)^2 + \\ &\qquad s_1(1-\nu^2)(\omega/\omega_0)^2(h/a)^2 \\ &A_7 = -5(1-\nu)(\alpha_3 + \alpha_4)(a/c)/12 \\ &A_8 = -5(1-\nu)(a/c)^2\alpha_4/12 + s_2(1-\nu^2)(\omega/\omega_0)^2(h/a)^2 \\ &A_9 = -5(1-\nu)(\alpha_3 - \alpha_4/2r_h)(a/c)/12 + s_3(1-\nu^2)(\omega/\omega_0)^2(h/a)^2 \\ &\qquad \omega_0^{\,2} = Eh^2/(\rho a^4) \quad\text{and}\quad \alpha_5 = \alpha_1 + 2\alpha_3 + \alpha_4 \end{aligned} \tag{7}$$

The system of Eqs. (6) can be further reduced to

$$\begin{aligned} &\nabla^2 W + A_1 W + A_2\Phi + A_3\Gamma = 0 \\ &A_4\nabla^2 W + A_5\Phi + \nabla^2\Gamma + A_6\Gamma = 0 \\ &A_7\nabla^2 W + \nabla^2\Phi + A_8\Phi + A_9\Gamma = 0 \end{aligned} \tag{8}$$

The solution of the differential Eqs. (8) can be expressed in terms of Bessel functions for the sandwich plate with no hole. Thus

$$\begin{aligned} W &= \sum_{j=1}^{3} L_j J_0(\lambda_j x) \\ \Phi &= \sum_{j=1}^{3} L_j \xi_j J_0(\lambda_j x) \\ \Gamma &= \sum_{j=1}^{3} L_j \eta_j J_0(\lambda_j x) \end{aligned} \tag{9}$$

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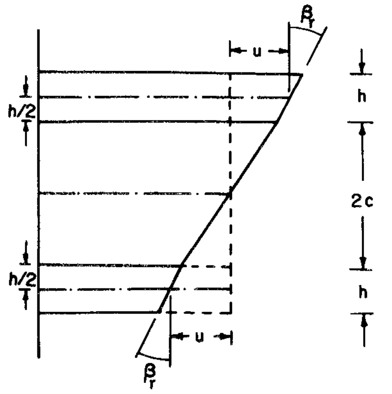

Fig. 1 Variation of radial displacement along the thickness of plate element.

In Eqs. (9) λ_j^2 $(j = 1,2,3)$ are the roots of the characteristic polynomial

$$\lambda_j^6 + a_1\lambda_j^4 + a_2\lambda_j^2 + a_3 = 0 \tag{10}$$

where

$$a_1 = A_3A_4 + A_2A_7 - A_1 - A_6 - A_8$$
$$a_2 = A_1(A_6 + A_8) + A_6A_8 - A_5A_9 + A_2(A_4A_9 - A_6A_7) + A_3(A_5A_7 - A_4A_8)$$
$$a_3 = A_1(A_5A_9 - A_6A_8)$$

Also, the quantities ξ_j and η_j are

$$\begin{aligned}\xi_j &= [(A_3A_7 - A_9)\lambda_j^2 + A_1A_9]/(-A_3\lambda_j^2 + A_3A_8 - A_2A_9)\\ \eta_j &= [-\lambda_j^4 + (A_1 + A_8 - A_2A_7)\lambda_j^2 - A_1A_8]/(-A_3\lambda_j^2 + A_3A_8 - A_2A_9)\end{aligned} \tag{11}$$

The expressions for U and V are now obtained with the help of Eqs. (6) and (9)

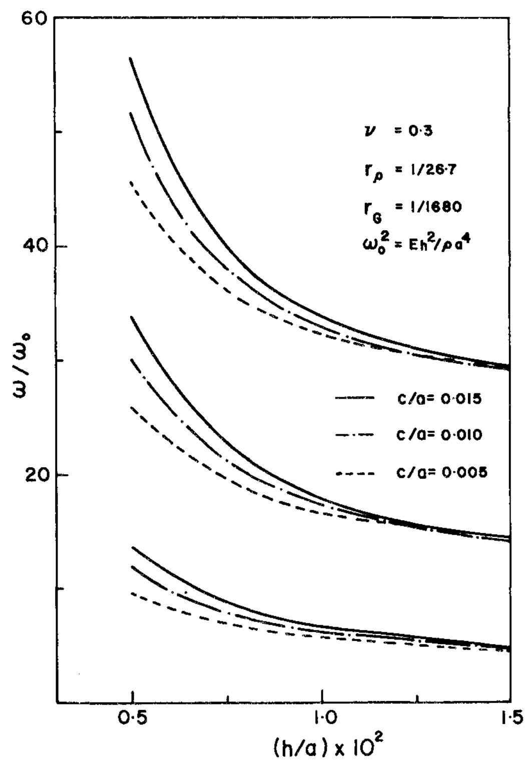

Fig. 2 Plot of nondimensional frequency ω/ω_0 vs h/a of the sandwich plate for constant c/a values.

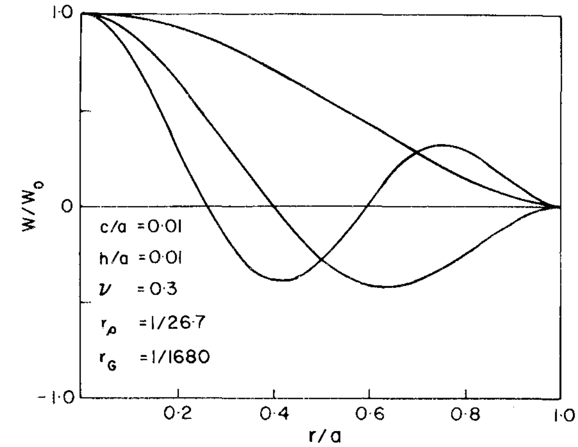

Fig. 3 Mode shapes for the clamped-circular sandwich plate.

$$\begin{aligned}U &= \sum_{j=1}^{3} Z_j W_{,x}\\ V &= \sum_{j=1}^{3} p_j W_{,x}\end{aligned} \tag{12}$$

Here, Z_j and p_j are given as

$$Z_j = (A_9\eta_j - A_6\xi_j + A_4A_9 - A_6A_7)/(A_6A_8 - A_5A_9)$$
$$p_j = (A_5\xi_j - A_8\eta_j + A_5A_7 - A_4A_8)/(A_6A_8 - A_5A_9)$$

The frequency equation can be generated from the three boundary conditions prescribed at the edge $x = 1 (r = a)$. For the clamped edge condition

$$U = V = W = 0 \quad \text{at} \quad x = 1 \tag{13}$$

Use of Eqs. (9) and (12), along with the abovementioned conditions yields the following determinantal equation

$$f(\lambda_j, \omega/\omega_0) = \begin{vmatrix} J_0(\lambda_1) & I_0(\lambda_2) & I_0(\lambda_3) \\ -Z_1\lambda_1J_1(\lambda_1) & Z_2\lambda_2I_1(\lambda_2) & Z_3\lambda_3I_1(\lambda_3) \\ -p_1\lambda_1J_1(\lambda_1) & p_2\lambda_2I_1(\lambda_2) & p_3\lambda_3I_1(\lambda_3) \end{vmatrix} = 0 \tag{14}$$

The presence of modified Bessel functions in place of Bessel functions are due to the fact that two roots λ_2^2 and λ_3^2 of the characteristic Eq. (10) are negative quantities for the prescribed values of the geometric and elastic parameters of the sandwich plate.

Results and Conclusion

Numerical results for clamped edge circular plates have been presented in this Note. The results for other boundary conditions can be generated without difficulty. In the case of plates having concentric hole at the center, the solution given in Eqs. (9) should be written in terms of Bessel function of the second kind with three more arbitrary constants. These additional constants are evaluated with the help of extra conditions at the inner boundary.

The natural frequencies for different modes are obtained from the frequency Eq. (14). As a result of highly transcendental nature of this equation, the roots cannot be obtained in a closed form. Numerical computations have been performed on a high-speed digital computer for a sandwich plate made of aluminum face sheets and aluminum honeycomb as the core material for which $\nu = 0.3$, $r_\rho = 1/26.7$, $r_G = 1/1680$.

The dimensionless frequency parameter ω/ω_0 is represented in Fig. 2 as a function of h/a and for three different values of c/a. It is interesting to note the effect of c/a on the frequency. For very thin face sheets, as c/a increases the frequency parameter increases. This variation in ω/ω_0 values for various c/a and a constant h/a gradually becomes smaller for thicker face sheets. This is because for thin face sheets, bending plays a minor role while in the case of thick face sheets flexure is significant. As mentioned earlier in this Note, it is assumed that the core takes only transverse shear and in the analysis the flexural rigidities of the face sheets about their own midplanes have been included.

Results for ω/ω_0 were again computed by omitting the effect of the rotary inertia from the governing differential equations. These results can be generated by redefining the four coefficients A_5, A_6, A_8, and A_9 in Eqs. (7) and then computing the roots of Eq. (14) again. For lower frequencies, the values obtained for ω/ω_0 by dropping the rotary inertia terms and the corresponding values shown in Fig. 2 are essentially the same. This conclusion is in agreement with the one for rectangular sandwich plates[5] and for homogeneous circular plates.[1] Rotary inertia terms would, however, play significant role for higher modes.

Figure 3 shows modal shapes for first three modes for which the frequency distribution is given in Fig. 2.

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Dependence of Plate-Bending Finite Element Deflections and Eigenvalues on Poisson's Ratio

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FOR uniform plates subjected only to a normal load and having boundaries consisting entirely of segments that are either a) clamped or b) simply supported and straight, the differential equations and boundary conditions depend on Poisson's ratio (ν) only indirectly (through the plate rigidity D). Hence, exact solutions for displacements and natural frequencies depend on Poisson's ratio only through D.[1] Cowper et al.[2] prove that finite element deflections and natural frequencies with conforming elements also depend on ν only through D. For nonconforming elements, however, the finite element approximations depend directly on ν even for the aforementioned class of problems of plate bending. No data are presently available in the literature on the extent of this dependence.

To study the effect of direct dependence on Poisson's ratio in the finite element approximations using nonconforming elements, numerical experiments were conducted on the static and free vibration analyses of a square plate with clamped and simply supported edges (see Fig. 1). Because of symmetry on the x- and y-axes, only one-quarter of the plate (shaded portion in Fig. 1a) is analyzed. The "P" and "Q" arrangements of mesh idealizations of the shaded one-quarter of plate are shown in Figs. 1b and 1c. Two problems of static analysis, viz., central concentrated load P and uniformly distributed load q on clamped square plate and one problem of free vibration analysis of simply supported square plate are worked out. The nonconforming triangular element of Narayanaswami[3,4] was used in the study. This noncompatible element has 18 degrees of freedom involving lateral displacement w and the two rotations, α and β, at the three corners and the three midside points. The interpolation function used is a quintic polynomial in x and y. Six values of Poisson's ratio were considered in the static analysis, viz., 0.0, 0.10, 0.20, 0.30, 0.40, and 0.495; and two values, viz., 0.0 and 0.3 in the free vibration analysis.

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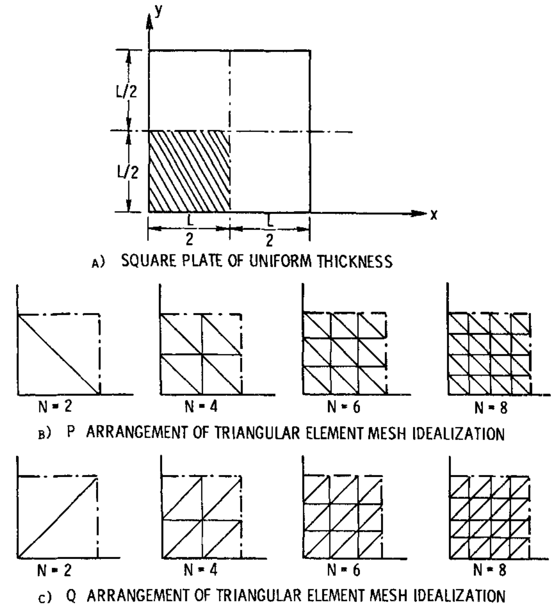

Fig. 1 Finite element idealization of square plate.

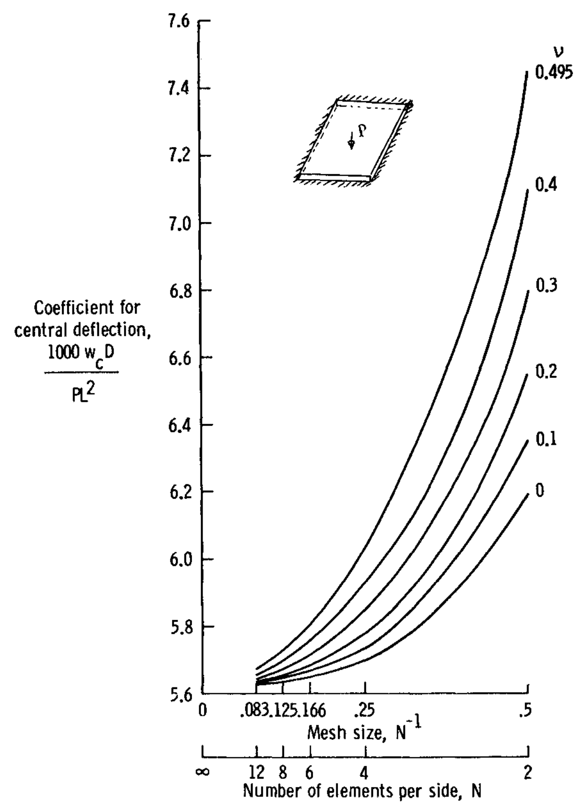

Fig. 2 Effect of Poisson's Ratio on nonconforming quintic Element.